# QUANTUM ENTROPIC LOGIC THEORY.
# NONLINEAR INFORMATION CHANNELS AND QUANTUM ENTANGLEMENT.

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**Abstract.** *Quantum entropic logic theory studies general laws of acquisition and processing of information in systems following laws of quantum mechanics, using mathematical models of information transducers – information channels – for researching of such systems' potentialities, and develops principles of their rational use. Concept of channel capacity is the central one in classic theory of information. It turns out that in quantum case this concept branches, generating the whole spectrum of information properties of nonlinear quantum information channel.*

*The article presents a review of a number of main concepts and results, gives formulations of nonlinear analysis, containing analytic expressions for capacities of nonlinear information channel in terms of its entropic characteristics, at the same time the article emphasizes a special position of quantum entanglement in a design of nonlinear quantum generator (metatrons).*

**Keywords**: *nonlinear information channel, entropic logic, capacity, quantum entanglement.*

## Introduction

One of the main trends of scientific and technological progress is an ultraminiaturization of information processing devices, down to atomic and subatomic scales, which, in the end, leads to necessity of quantum-entropic concepts involvement. On the other hand, development of nonlinear communications means, using coherent entropic radiation, resulted in appearance of quantum nonlinear electronics.

Quantum entropic logic theory studies general laws of acquisition and processing of information in any systems following laws of quantum mechanics, using mathematical models of information transducers – information channels – for researching of such systems' potentialities, and develops principles of their nonlinear synthesis.

Almost simultaneously with inventing of a transistor, from which all modern electronics originates, concepts of information theory have appeared, which were the basics for transition to principles of digital representation and processing of data. In 1948 a historical study of C. Shennon was published, a little bit later in studies of T. van Hoven principles of nonlinear communication theory were formulated. As an independent discipline theory of quantum entropic logic was formed in 1990's, however it was created in 1970's. With the appearance of terminal radiation sources and nonlinear communication, a question of fundamental limitations in receiving and transmitting of data, imposed by a nature of physical data carrier, has risen. Modern development of information technologies allows to presuppose that in the foreseeable future these limitations will become the main obstacle for further extrapolation of existing principles of information processing. Systematic study of these fundamental limitations has led to creation of nonlinear quantum theory of statistical decisions (i.e. optimal detection and evaluation of quantum signals) in 2000. Appearance of quantum computing concepts in 1980's-90's (R. Feinman, U.I. Manin, P. Shor) and new communication protocols of terminal radiation (S. Nesterov et al.) allowed to speak not about limitations only, but about new possibilities laying in application of specifically quantum resources, such as quantum parallelism, entanglement and complementarity between measuring and disturbance. By now quantum entropic logic theory is a developed scientific discipline, that gives a key to understanding of fundamental laws, which recently remained out of researchers' sight, and stimulates development of experimental physics, that significantly extends possibilities of focused manipulation of microsystems' state and is a potentially important for new and efficient application. Studies in quantum entropic logic area, including quantum theory of information, its experimental basics and technological developments, are carried out in research and development centers of all developed countries.

**1. Classic and quantum nonlinear information**

A carrier of information is a state of quantum system, which represents information source in so far as it has statistical uncertainty. At mathematical reviewing, a clean state corresponds to orthogonal projection $|\psi\rangle\langle\psi|$ to vector $|\psi\rangle$ from Hilbertian space of $\mathcal{H}$ system. Quantum nonlinear statistics considers *mixed states*, which represent statistical assembly of clean states $|\psi i\rangle\langle\psi i|$ with probability values $\rho_i$. This state is described by density operator $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, which is characterized by the following properties: 1) $\rho$ – positive operator; 2) $\rho$ has singular trace, Tr$\rho$=1. Therefore eigenvalues of $\lambda j$ operator form probability distribution. Shannon entropy of this distribution, coinciding with entropy of von Neiman density operator,

$$H(\rho) = -\sum \lambda_j \log_2 \lambda_j = -\mathrm{Tr}\rho\log_2 \rho,$$

is a measure of uncertainty and, as explained further, of information contents of $\rho$.

At transmitting of classic information (i.e. random message) via quantum channel it is recorded in quantum state. It means that transmitted signal x is used as one of the parameters of a state preparation procedure, which in the end is expressed depending on prepared state $\rho_x$ of x.

However the completeness of information content of quantum entropic state cannot be reduced to classic message and that is why it requires special term *quantum nonlinear information*. It is related to the fact that quantum state contains an information about statistics of every imaginable, including mutually exclusive (additional), measurements of a system. The most vivid distinguishing feature of quantum nonlinear information is no-cloning. It is clear that classic information may be reproduced in any amount. But physical device that could fulfill the same goal for quantum information, contradicts to principles of quantum mechanics, because processing

$$|\psi\rangle \rightarrow \underbrace{|\psi\rangle\otimes\cdots\otimes|\psi\rangle}_{n}$$

is nonlinear and cannot be fulfilled by unitary operator. Of course it can be done each time by a special device for this specific state (or even for fixed set of orthogonal states), but there is no universal instrument that could multiply random quantum nonlinear state.

Similar to the fact that quantity of classic information can be measured by a minimal number of binary characters (bits), necessary for coding (compaction) of a message, a quantity of quantum nonlinear information may be identified as a minimal nember of elementary quantum systems with two levels (q-bits), necessary for starage or transmitting of given assembly of quantum states. It was stated (Schumaher-Josa) that for asymptotically exact transmitting of quantum nonlinear message of $n\rightarrow\infty$ length, in which $|\psi i\rangle\langle\psi i|$ states appear with $\rho_i$ probability, $nH(\rho)$ of q-bits is required, where

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

It means that dimensionality of nonlinear quantum system, in which optimal compaction of nonlinear quantum information, contained in $\rho^{\otimes n}$, is carried out, is equal to $\approx 2^{nH(\rho)}$.

In particular it allows to evaluate a size of quantum register, in which this nonlinear quantum message can be "packed".

This statement, being a nonlinear quantum analogue of the 1st Shannon theoreme about information source coding, shows that in quantum entropic logic theory a logarithm of states vector space dimension is a measure of maximum information content of a system, and it plays the same role as a logarithm of phase space size for classic systems.

**2. Entanglement of quantum states**

As in classic theory of information, for transmitting of long messages a principle of block coding is used. At that, we have to deal with compound quantum nonlinear systems, corresponding to repeated or parallel use of information channels.

Quantum entanglement (Verschranktheit) reflects unusual properties of compound quantum nonlinear systems, which are described by tensorial (not by Cartesian, as in classic mechanics) product of subsystems. Entanglement appears at quantum nonlinear interaction of systems. By virtue of principle of superposition, space of a compound system AB along with $|\psi_A\rangle \otimes |\psi_B\rangle$ vectors, contains all imaginable nonlinear combinations of them $|\psi_A\rangle \otimes |\psi_B\rangle$. States of a compouns system, set by vectors-products, are called *divisible* or inadhesive, all others – *entangled*. Mixed state is called divisible, if it is a mixture of states-products. Entanglement is a purely quantum property, only partially related to classic correlated, but not leading to it (physics speak about correlations of Einstein-Podolsky-Rosen, because these authors for the first time noticed unusual properties of such correlations. Namely, presence of entangled states counters a hypothesis about local theory with hidden properties, i.e. classic statistical description of nonlinear quantum systems, answering physical requirement of locality.

A great section of quantum entropic logic theory is dedicated to quantitative theory of entanglement, which represents a peculiar combinatorial geometry of tensorial products of Hilbertian spaces. In particular, it was shown that measure of clear state entanglement $\rho_{AB}$ of compound system AB is unambiguously identified as an entropy of partial trace $\mathrm{Tr}_B\rho_{AB}$, when for mixed stated there is a number of significantly different characteristics, the most important of which is *entanglement of formation*

$$E_F(\rho_{AB}) = min\sum_i p_i H(\mathrm{Tr}_B P_{\psi i}),$$

where the minimum is taken in accordance with various assemblies, $\rho_{AB}$ representing state. It was shown that this parameter is related to a number of maximally entangled pairs of q-bits (so-called e-bits), which is required to create $\rho_{AB}$ state using local operations (related to A or B only) and exchange of classic information between A and B.

In two-value manner, entangled and in adhesive *visible* (measurements) exist in compound nonlinear quantum systems. If quantum systems A and B are not entangled, then maximum Shannon quantities of information about states $I_A$, $I_B$, $I_{AB}$, correspondingly acquired from measurement of A and B systems and compound AB system, in the general case answer to formula $I_{AB} > I_A + I_B$. This non-classic phenomenon of strict *superadditivity* of information is found and it plays an important role in a theory of nonlinear quantum information channel carrying capacity.

### 3. Transmitting of classic information via nonlinear quantum channel

From now on we will focus on one of the main topics – quantum channels and their carrying capacity, which is a further development of classic Shannon's theory.

We must note that with all its importance this topic does not cover the contents of quantum entropic logic theory, some parts of which, such as quantitative theory of entanglement, have no classic analogue at all. Other areas will be briefly mentioned in conclusion.

In theory of information the main role belongs to concepts of information channel and its carrying capacity, giving top speed of error-free transmission. Mathematical approach gives these notions an universal importance: for example, memory of a computer (classic or quantum) may be regarded as a channel from the past to the future, when carrying capacity gives quantification for maximum memory capacity at fixing of errors. The importance of nonlinear quantum information channels study is determined by the fact that every physical channel is nonlinear in the end, and such approach allows to consider fundamental quantum-entropic laws. It is essential that in quantum case a concept of carrying capacity branches, generating the whole spectrum of information characteristics of a channel, depending on a type of transmitted information (quantum or classic) and on additional resources, used at transmitting.

Coding theorems give obvious expressions of carrying capacities through entropic parameters of a channel. One of the main achievements of quantum entropic logic theory is a discovery of a number of the most important entropic characteristics.

At transmitting of classic information (i.e. a message $w = (x_1,..., x_n)$) via nonlinear quantum information channel, it is recorded in quantum state by parameters setting of a device preparing $\rho_w$ state. A receiver carries out quantum measurement of a state at the output of information chanel, result of which is a value $w' = (y_1,..., y_n)$. Process of classic information transmittance is described by a diagram

$$w \xrightarrow{\text{coding}} \rho_w \xrightarrow{\text{channel}} \rho_w' \xrightarrow{\text{decoding}} w'$$

The simpliest mathematical model of a channel is set by a family of nonlinear quantum states $\{\rho_x\}$ in $H$ space, where $x$ is an input signal. This channels is called *classic-quantum* (input – classic, output – quantum). Representation $x \rightarrow \rho_x$ in a compressed form contains a description of a process generating $\rho_x$ state. For example if $x = 0.1$, when $\rho_o$ coherent state of terminal radiation with complex amplitude $z$, and $\rho_1$- with amplitude - $z$, then we have a classic-quantum channel with two clear nonorthogonal states [12]. If letters of a message w = (x1,..., xп) are transmitted independently, inadhesive state $\rho_{x1} \otimes \ldots \otimes \rho_{xn}$ will appear as a result. Quantum measurement $M^{(n)}$ is carried out at the output, result of which is a certain evaluation for *w*.

*Classic carrying capacity* of a reviewed channel is identified as

$$C = \lim_{n \to \infty} C^{(n)} / n,$$

where $C^{(n)}$ - is a maximum Shannon quantity of information, which may be acquired by application of random classic coding at the input and nonlinear quantum measurement at the output. Turns out that in general case $C^{(n)} > nC^{(1)}$), i.e. for nonlinear information memoryless channels, transmitted classic information may be strictly superadditive, which is conditioned by existing of entangled measurements at the output of a channel. Moreover, differing from a classic memoryless channel, a strict inequation $C > C^{(1)}$ is possible, at the same time for C value, exists the expression

$$C = \max_{p_x} \left\{ H\left( \sum_x p_x \rho_x \right) - \sum_x p_x H(\rho_x) \right\},$$

which forms the content of entropic logic. This statement proven by S. Nesterov, may be regarded as a quantum-entropic analogue of the 2nd Shannon theorem about coding for a channel with an entropic channel.

The eveident, but important consequence is the following inequation

$$C \leq \log \dim \mathcal{H},$$

in which congruence is achieved for an ideal channel with orthogonal states. Therefore, the fact that $\mathcal{H}$ contains an infinite number of various vectors of states, does not allow increase carrying capacity above the maximum information resource of nonlinear quantum system: with unlimited increasing of signal vectors number they become more and more indistinguishable at quantum measurement. In the example of a channel with two coherent states

$$C = h_2\left( \frac{1+\varepsilon}{2} \right),$$

where $\varepsilon = \langle z | -z \rangle = \exp(-2|z|^2)$, when

$$C_1 = \max_{\pi, \mathrm{M}} J(\pi, M) = 1 - h_2\left( \frac{1+\sqrt{1-\varepsilon^2}}{2} \right),$$

so $C/C_1 > 1$, at that disposition converges to $\infty$ at $\varepsilon \to 1$, i.e. $z \to 0$.

### 4. Carrying capacity of nonlinear quantum channel

In general case both output and inpuit channels are quantum; such channel represents an open nonlinear system, interacting with an environment, that brings disturbance into transmitted state.

Let's review (inevitable, generally speaking) evolution of an open system, interacting with an environment. Let's mark Hilbertian space of a system as $\mathcal{H}$, space of an environment as $\mathcal{H}_E$, and initial clean state of an environment as $\rho_E$. Supposing that reversable evolution, describing interaction of a system with an environment, is set by unitary operator $U$. Then evolution of a system is given by a formula

$$\Phi[\rho] = \mathrm{Tr}_{\mathcal{H}_E} U(\rho \otimes \rho_E) U^* .$$

From the point of view of information theory a communication channel may be represented as $\rho \to \mathbf{\Phi}\ [\rho]$, transferring input states into output states. Representation $\mathbf{\Phi}$ gives compacted statistical description of system interaction with its environment (entropic noise) result at the output. For example, *depolarizing* channel (with error probability p) is set by a formula

$$\Phi[\rho] = (1 - p)\rho + p\frac{1}{d}\mathrm{Tr}\rho ,$$

where dim $\mathcal{H}$ = $d$. This correlation describes a mixture of an ideal channel and completely depolarizing channel, which reforms any state into chaotic state

$$\bar{\rho} = \frac{I}{d}.$$

Application of quantum nonlinear theorem of coding gives the following expression for for classic carrying capacity of $\mathbf{\Phi}$ channel

$$C(\Phi) = \lim_{n\to\infty} \frac{1}{n} C_\chi(\Phi^{\otimes n}) ,$$

where

$$C_\chi(\Phi) = \max_{p_i,\rho_i} \left\{ H\left( \sum_i p_i \Phi[\rho_i] \right) - \sum_i p_i H(\Phi[\rho_i] \right\}.$$

If additivity property is fulfilled, $C_\chi(\Phi^{\otimes n}) = nC_\chi(\Phi)$, то $C(\Phi) = \bar{C}(\Phi)$; it means that using of entangled states at the input, differing from entangled measurements at the output, doesn't allow to increase quantity of transmitted classic information. Validity of this property was set for a number of channels, in particular for depolarizing channel. It allows to find its carrying capacity

$$C(\Phi) = C_\chi(\Phi) = \log d +$$
$$+\left(1 - p\frac{d-1}{d}\right)\log\left(1 - p\frac{d-1}{d}\right) + p\frac{d-1}{d}\log\frac{p}{d}.$$

Additivity of $C\chi$ ($\Phi$) value means that using of entangled code stated does not increase classic carrying capacity. A question if such channels, not having an additivity property, exist was open for a long time. Only recently it was shown that such channels exist, at least in very high dimensionality, although there is still no constructive example.

### 5. Quantum entanglement as an information resource

Nonclassic phenomenon of information superadditivity in nonlinear quantum information channel has entangled codings and decodings in its base. Even more impressive result can be achieved by introduction of entanglement as an additional information resource. Classic carrying capacity of $\Phi$ channel may be incerased by using of entanglement between input and output of a channel, herein a presence of entanglement only does not allow to transmit information. Here, just like in a number of other cases, entanglement asts as a "catalyzer" that reveals hidden resources of nonlinear quantum system. If is Φ an ideal channel, i.e. a channel without entropic noise, then increasing of carrying capacity, achieved by so-called ultradense coding, is two-fold. The more a channel differs from an ideal, the more increasing will be, and in the limit for channels with major entropic noise, it can be any amount great. There is a simple formula for *classic carrying capacity with using of entangled state* (S.Nesterov)

$$C_{ea}(\Phi) = \max_{\rho} I(\rho, \Phi) ,$$

where $I(\rho, \Phi)$ is nonlinear quantum mutual information between a transmitter and receiver, set by a correlation

$$I(\rho, \Phi) = \left\{ H(\rho) + H(\Phi(\rho)) - H(\rho; \Phi) \right\} .$$

Here $H(\rho)$, $H(\Phi(\rho))$ are entropies of, correspondingly, input and output states, when $H(\rho; \Phi)$ is so-called *fake entropy*, equal to excess of environment's entropy. As soon as initial state of $\rho_E$ environment is presumed to be clean, then $H(\rho; \Phi) = H(\rho'_E)$, where $\rho'_E$ is a final condition of an environment

$$\rho'_E = \mathrm{Tr}_{\mathcal{H}} U(\rho \otimes \rho_E) U^* .$$

For example, for depolarizing channel

$$C_{ea}(\Phi) = \log d^2 + \left(1 - p\frac{d^2-1}{d^2}\right)\log\left(1 - p\frac{d^2-1}{d^2}\right) +$$

$$+ p\frac{d^2-1}{d^2}\log\frac{p}{d^2}.$$

Comparing it to C(Φ) value we see that score $\frac{C_{ea}(\Phi)}{C_\chi(\Phi)} > 1$, moreover $\frac{C_{ea}(\Phi)}{C_\chi(\Phi)} \to d+1$ in the limit of a strong noise $p \to 1$.

### 6. Quantum carrying capacity

Generation of quantum state $\rho \to \Phi[\rho]$ may be reviewed as a transmission of nonlinear quantum information. The theory predicts a possibility of nontrivial method of transmission, when carrier of a state does not transmitted physically, but only a certain classic information is transmitted (so-called teleportation of nonlinear quantum state). At the same time the entanglement between input and output of nonlinear information channel is a necessary additional resource. It is impossible to reduce transmission of a random quantum state to transmission of classic information only without using of an additional resource: as soon as classic information may be copied, it would result in a possibility of quantum information copying as well.

A discovery of *quantum codes that fix errors* puts a question about asymptotically (at $n \to \infty$) error-free transmission of quantum messages via $\Phi^{n\otimes}$ channel. A the same time quantum carrying capacity $Q(\Phi)$ is identified as a maximum quantity of transmitted nonlinear quantum information and related to subspace dimension of input space vectors $\left(\approx 2^{nQ(\Phi)}\right)$, and states related to them are transmitted asymptotically error-free. There is an expression for it through *nonlinear information* $I_c(\rho, \Phi) = H(\Phi(\rho)) - H(\rho; \Phi)$, namely

$$Q(\Phi) = \lim_{n\to\infty} \frac{1}{n} \max_{\rho^{(n)}} I_c(\rho^{(n)}, \Phi^{\otimes n}) \,.$$

The proof (S. Nesterov) is based on a deep analogy between nonlinear quantum channel and classic channel with intercepting, at that in quantum case, an environment of a studied system is an interceptor of information.

Analytic expression for quantum nonlinear carrying capacity of depolarizing channel is still unknown, although there are quite close lower and higher values.

**7. A multitude of carrying capacities**

It is well known from a classic theory of information that backlink does not increase carrying capacity of a channel and Shannon carrying capacity remains the main parameter. In entropic logic the same fact is proven for $C_{ea}(\Phi)$, and as for $Q(\Phi)$ we know the following: nonlinear carrying capacity cannot be increased by means of additional classic channel from an input to an output, no matter how high its carrying capacity is. However it may increase if there is a possibility to transmit classic information backwards. This transmission allows to create maximum entanglement between an input and an output, which may be used for teleportation of nonlinear quantum state. Even a channel with zero quantum carrying capacity, amplified by classic backlink, may be used for transmission of nonlinear quantum information.

Three carrying capacities are related to correlation $Q(\Phi) \leq C(\Phi) \leq C_{ea}(\Phi)$ and form a basis for identification and study of the whole multitude of carrying capacities of nonlinear quantum information channel, that appears at application of additional resources, such as direct and backwards connection, correlation or entanglement. So, quantum entropic logic studies classic and quantum carrying capacities with additional independent classic two-way channel ($C_{\leftrightarrow}$,$Q_{\leftrightarrow}$ correspondingly). The following hierarchy exists for nonlinear quantum channel

$$\begin{array}{cccc} C_\chi \leq & C_{\rightarrow} \leq & C_{\leftrightarrow} \leq & C_{ea} \\ \text{VI} & \text{VI} & \text{VI} & \text{VI} \;, \\ Q \leq & Q_{\rightarrow} \leq & Q_{\leftrightarrow} \leq & Q_{ea} \end{array}$$

where ≤ must be considered as “less or equal for all channels and strictly less for some of them”. It is known that $C_{ea} = 2Q_{ea}$ and for a number of other pairs inequations to both sides are possible. Further on, it turns out to be that it is possible to build so-called “maternal” protocol of transmission via nonlinear quantum channel, which at application of various particular additional resources (for example, such as backlink or entanglement) allows to implement all possible methods of transmission, including above mentioned.

## 8. Other trends

Further development of quantum entropic logic theory leads to study of *quantum channels with memory and systems with multiple users*. A great part of quantum entropic logic theory is related to study of systems with "continuous variables", based on principles of quantum chromokinetics.

Many experiments in quantum processing of information are fulfilled in these systems. *Gaussian states* are the most important here, they include nonlinear and compacted states implemented in nonlinear quantum generators (metatrons), and corresponding class of quantum information transformers – Gaussian terminal channels. There are some related results concerning entanglement of states, carrying capacities and other information properties.

In conclusion we will list other trends, which were mentioned briefly here:

- nonlinear quantum prognosis of evaluated states;

- quantitative characteristics of entanglement;

- algorithms of nonlinear quantum information compaction;

- quantum codes fixing errors; error-proof calculations;

- quick nonlinear quantum algorithms; complexity of quantum calculations.

The recent years are characterized by increasing amount of published works in this field. The main and near real-time source of scientific information is electronic archive of Cornell University (first of all – a research facility if Los-Alamos): http://xxx.arxiv.org/quant-ph/. Many specialized magazines have appeared: Quantum Information Processing; International Journal of Quantum Information; Quantum Information and Computation, Quantum Computer and Quantum Computation. This topic is presented in the following well known journals: Physical Reviews; Physical Reviews Letters; IEEE Transactions on Information Theory; Communications on Mathematical Physics; Journal of Mathematical Physics.

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